Dynamic Nash Equilibrium in an Overlapping-Generations Model:

A Methodological Exercise in Economic Simulation*

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ABSTRACT

Household life-cycle decisions are fundamental to policy issues ranging from regulation to monetary policy and fiscal entitlements. These life-cycle economics relate to efforts by the U.S. Department of Homeland Security to better forecast how hypothetical disruptions or "scares" might affect financial markets and economic equilibrium. We propose incorporating agent-based simulation as a tool for exploring such questions. We present a methodology for using simulation in conjunction with conventional economic theory, and conduct an exercise to demonstrate that methodology. Specifically, we present a life-cycle model of overlapping generations in a closed economy with labor, goods, and banking markets. We then present an agent-based computer simulation in which firms use simple search algorithms to optimize employment and prices. The simulation converges to its analytically derived Nash equilibrium, thereby verifying that the simulation is consistent with theory, and validating the theory for the decision rules and interaction protocols defined within the agent simulation.

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1 Introduction

The national homeland-security agenda includes achieving a comprehensive understanding of the potential economic effects of terrorist acts. In addition to the direct impacts of an attack, such events might also incite overreactions by firms and households to the detriment of the economy. These secondary effects relate to a broader set of issues concerning the influence of information and perception on economic decision makers. For example, one might ask "How might households alter their consumption rates in response to a terrorist act?", and "How might those decisions affect the pricing and employment decisions of firms?" The conventional interplay between rigorous theory and statistical empirics provide insight into these questions, but often focuses on the properties of equilibria, without exploring conditions under which a system will or will not converge from a disequilibrium state to an "expected" equilibrium. We propose agent-based simulation as a viable extension to conventional methods, allowing for the adoption of established economic principles to provide a verifiable foundation for exploratory economic models.

One concern posed by terrorism scenarios is their potential impact on financial markets. Although there exists a growing literature on agent-based financial trading (e.g. Domowitz and Wang 1994, Lettau 1997, Farmer 2001, and Farmer et al 2003), the financial impacts of terrorist scenarios require models of a broader set of interdependent markets. We present such a model, in which firms and households simultaneously participate in labor, goods, and banking markets.

2 Purpose and Scope

This article is arranged as follows. First, we present some basic methodology for synthesizing theory and simulation, describe the potential benefits from exercises that use this methodology, and suggest assessment criteria for evaluating such exercises. Second, we present an analytical model of oligopoly in an economy of overlapping generations, in which a subset of variables, called parameters, are held constant. We derive dynamic Nash equilibrium prices, expressed in the abstract in terms of functions of variables. We specify arbitrary parametric values in order to calculate an instance of the closed-form equilibrium solution. Third, we describe a multi-agent simulation of the prescribed model. The simulation is calculated using the same parametric values used previously for calculating the equilibrium in the analytical model. We present the simulation outcome, expressed as the convergence of the simulation variables, and show that it is consistent with the analytically derived outcome. Fourth, having verified the baseline simulation, we suggest some optional extensions for exploring current problems in economics.

3 Methodology

Agent-based simulation has yet to gain wide acceptance in the economics community. The delay might stem, in part, from the fact that desktop object-oriented programming technologies are relatively young and require a substantial human-capital investment for researchers to become skilled in their use. However, a more fundamental issue hindering the adoption of simulation arises when the agent community designs artificial economic models out of computational building blocks without providing the economics community with a familiar theoretical blueprint of the economic model as a whole.

To clarify, conventional economic theory is derived from a set of "accepted" axioms regarding the preferences, objectives, and strategies of individual economic actors. A theory takes the form of a formal mathematical model (a.k.a. analytical model, theoretical model, formal theory, and rigorous theory) in which axioms beget propositions, which are supported by proofs. Such models allow for closed-form solutions (a.k.a. analytical solutions, analytical results, and closed-form results) and comparative-static analyses. A critique from the agent community is that this approach does not explore the often complex interactions that are assumed to move a real-world system toward its equilibrium. In other words, analytical solutions add little value without validation against "realistic" processes by which individuals obtain and process information, make decisions, and interact.

In contrast, an agent-based model takes the form of a simulation in which axioms beget a set of independent computer programs (agents), which beget a procedure by which the agents make decisions and interact through a sequence of time-steps. A simulation is said to *correspond to* a formal theory if the variables and objectives of the agents correspond to those defined in the theory. For example, consider a theoretical model consisting of firms seeking to maximize profit, a function of cost and revenue. A corresponding simulation would include agents containing cost, revenue, and profit variables, as well as decision and interaction rules by which each agent seeks to maximize its profit variable. A key observation is that there usually exist limitless possible decision rules and interaction rules allowing for profit maximizing agents. Thus, there usually exists a many-to-one correspondence of simulations to a given theory. A critique from theorists is that such simulations can be informative, but they neither constitute nor allow for the sort of general propositions and proofs provided by mathematical analysis. In other words, such simulations add little value without verification against formal theory.

For clarity, one must distinguish a simulation from a calculation. A simulation corresponds to a specific set of decision and interaction rules. A decision rule can be as simple as an arithmetic function or as sophisticated as a genetic program. Decision rules are intended to model the decision and learning processes of economic actors. Interaction rules define the process whereby agents exchange information and conduct transactions. However, simulation results can depend not only on the decision and interaction rules, but also on parameters (e.g. initial and boundary conditions) and the sequence of decisions and interactions. Thus, let a *calculation* define an instance of a simulation pertaining to a particular set of parameters and sequence of decisions and interactions. There usually exists a many-to-one correspondence of calculations to a given simulation.

3.1 Consistency Exercises

In response to both of the previously-discussed critiques, we submit that simulation can be used in conjunction with formal theory, not only to verify simulation results, but also to validate and extend theoretical results. We propose an approach that begins with an analytical model and its analytically derived equilibrium solution. In subsequent steps, the researcher designs, builds, and executes a corresponding bottom-up simulation of the economic model, and compares the convergence of calculations against the analytical solution. If a calculation converges to the analytic solution, then the simulation is said to be *consistent* with theory. We call such a comparison a *consistency exercise*.[1]

In a strict sense, consistency implies that there exists at least one calculation for which the analytical solution holds, and for which the simulation corresponds to theory. It does not imply that all calculations corresponding to a given simulation will be consistent, and certainly does not imply that all simulations will be consistent. In lieu of these statements, one might mistakenly conclude that consistency exercises are of little importance, but closer inspection reveals that they can have profound importance when one considers their broader implications.

The first implication of consistency exercises pertains to establishing the robustness of theory in the context of economic processes, and discovering worthwhile revisions and extensions to theory. Consider that prior to a consistency exercise, an analytical solution can be proved in the abstract, but has yet to be demonstrated for even one sequence of decisions and interactions. It is therefore worthwhile to consider the following hypotheses:

1. a theory always holds for a simulation,
2. a theory sometimes holds for a simulation,
3. a theory never holds for a simulation,

and

4. a theory holds for all simulations,
5. a theory holds for some simulations,

[1] Our notion of consistency exercises stems from the suggestion that simulation can serve as a means to theoretical discovery (see Ostrom 1988, Gilbert and Terna 2000, Luna and Stefansson 2000, and McCain 2000, which are summarized in Hand et al 2005, Ch. 4). Additional rationales for the use of simulation relate to its potential for exploring models that are either extremely complex or have no closed-form solution, but these latter objectives are beyond the scope of this article.

6. a theory holds for no simulations.

We explore hypotheses 1-3 by executing many calculations corresponding to a single simulation. Although hypotheses 1 and 3 can never be proved, the objective is to determine whether the rules defined by a particular simulation tend to be consistent with theory. Key findings can arise when some calculations converge and others do not. Such cases can allow the investigator to identify parameters or path-dependent "tipping points" that lead to outcomes not predicted by theory.

We explore hypotheses 4-6 by changing the decision or interaction rules (processes). Although hypotheses 4 and 6 can never be proved, the objective is to determine which processes are consistent with theory. Key findings can arise when some simulations tend to be consistent with theory and others do not. Such cases can lead to worthwhile revisions or extensions to theoretical models.

In summary, we suggest a methodology by which investigators use consistency exercises to explore the robustness of theory in the context of economic processes. For completeness, the methodology requires some well-defined criteria for assessing each exercise. We suggest the assessment criteria in Table 1 for categorizing the success of consistency exercises.

Under these criteria, the exercise presented in the following sections achieves a *minimum* success rating by providing a simulation of an economy that converges to its dynamic Nash equilibrium. In short, such an exercise serves as an important validation point for the economics community.

In the broader context of validation, the methodology proposed here has important implications for practitioners. Currently, economic theories are typically validated through empirical analyses, which are often limited by degrees of freedom or data availability. When there is no means to validate theory through empirical observation, consistency exercises provide an intermediate step toward validation. By establishing consistency between a theory and a process, one focuses debate on whether the process is representative of reality.

The second implication of consistency exercises pertains to verifying the performance of simulations with respect to “accepted” theory. Consider that one of the most common goals in the use of simulation is to quickly determine the most likely outcomes given a set of user-defined parameters. By establishing consistency for a range of parameters, consistency exercises instill confidence that variations in a simulation’s parameters will result in consistent outcomes.

3.2 The Limits of This Article

We limit our investigation to a single calculation corresponding to a single simulation. The strength of conclusions that can be drawn from a single calculation is limited for reasons discussed above. However, the exercise provides an adequate example of the methodology.

We also limit the economic model itself. The analytical model in section 4 includes many simplifying assumptions, which could certainly be relaxed for a more general investigation. Furthermore, the simulation described in section 5 assumes decision rules that are arguably either too simplistic or to sophisticated in comparison to real-world decision makers. Nevertheless, the goal of this exercise is not to answer unresolved questions from life-cycle economics, but rather to demonstrate an instance of the methodology in the context of conventional economic principles, and thereby form a foundation for future research.

4 Analytical Model

Our model builds upon the considerable foundation of life-cycle economics stemming from the early work of Fisher (1930), Friedman (1957), Modigliani and Brumberg (1958), and Ando and Modigliani (1963), and the overlapping-generations models of Samuelson (1958), Wallace

(1980), Balasko et al. (1980-1981), and Tirole (1985). We model a discrete-time closed economy comprised of H households and F firms. Households decide how much to consume, borrow, and save each period. Firms decide whether to increase or decrease price and employment each period. Firms also act as passive lenders in a banking market by making their cash reserves available for loans to households. There is no money creation. For analytical convenience, wages, productivity rates, and interest rates are fixed, and marginal cost is constant and equal across firms.

Households grow older with each time period, and face a lifespan comprised of an employment-eligible (career) phase, during which households can earn wages in the labor market, and a retirement phase, during which households can only consume by withdrawing funds from their private savings. Households cannot substitute intertemporally by accumulating goods, but they can borrow funds from firms or deposit savings with firms via a banking market. Banking allows households to smooth their consumption patterns over their lifespans according to a conventional life-cycle hypothesis.

Each firm seeks to maximize short-run profit by hiring labor from households via the labor market, and producing goods to sell to households in the goods market. Firms earn *nominal* profits by charging prices above marginal cost and by charging interest on loans. Firms earn *real* profits by spending nominal profits to purchase back excess goods for their own consumption.

In this section, we derive general equations for the choice variables of households and firms. For households, we derive the optimal consumption expenditure and savings contribution for each period. For firms, we derive Nash equilibrium prices, where each firm's price is a function of its labor share and other firms' prices. We use these results in subsequent sections to calculate general equilibrium conditions. The analyses in this section utilize the constants in Table 2 and market variables in Table 3.

4.1 Households and Financial Optima

We now derive the household's desired consumption expenditure and banking transaction for each time period. Consumption must be non-negative, but a banking transaction can be either positive in the case of a deposit or negative in the case of a loan or withdraw. This analysis uses the household variables defined in Table 4.

Each household is comprised at any given time of a single individual who becomes employment-eligible at $Age_{\min}$, retires after Age_{retire}, and dies after $Age_{\max}$. Let Age_0 denote a household's current age measured in years, where $Age_{\min} \leq Age_0 \leq Age_{\max}$. Time is discrete, with a fixed number of λ periods per year. We define T_0 as the number of periods for consumption before the household expires, where

$$T_0 \equiv \lambda(Age_{\max} - Age_0) \begin{Bmatrix} > 1 & \forall\, Age_0 < Age_{\max} \\ = 1 & \forall\, Age_0 = Age_{\max} \end{Bmatrix}. \tag{1}$$

We define K_0 as the number of time-steps for earning income before the household retires, where

$$K_0 \equiv \lambda(Age_{retire} - Age_0) \begin{Bmatrix} > 1 & \forall\, Age_0 < Age_{retire} \\ = 1 & \forall\, Age_0 = Age_{retire} \\ = 0 & \forall\, Age_0 > Age_{retire} \end{Bmatrix}. \tag{2}$$

Any household that is employed by a firm supplies one unit of labor per period to its employer. All labor is supplied in discrete units, denoted $l_t \in \{\text{positive integers}\}$.

Each household derives utility in period *t* by consuming q_t units of goods. Utility is defined by $u_t = (q_t)^\beta$, where β ≡ consumption elasticity ∈ (0, 1), so that $u_t' > 0$ and $u_t'' < 0$. Each household valuates future consumption with respect to its internal discount rate, d_h, which implies that the current utility derived from expected future consumption is

$u_0 = \frac{u_t}{(1+d)^t} = \frac{(q_t)^\beta}{(1+d)^t}$, and

$$\frac{\partial u_0}{\partial q_t} = \beta \frac{(q_t)^{\beta-1}}{(1+d)^t}. \tag{3}$$

Households with a time-preference can increase utility by substituting consumption between time periods according to the following first-order condition, which must hold for any two time periods t_1 and t_2:

$$\frac{\partial u_0}{\partial q_{t_1}} = \frac{\partial u_0}{\partial q_{t_2}}. \tag{4}$$

Combining (3) and (4) yields $\frac{q_t}{q_0} = (1+d)^{\frac{t}{\beta-1}}$, which provides the ratio of future-to-current consumption expenditure:

$$\frac{c_t}{c_0} = \frac{p_t q_t}{p_0 q_0} = \frac{p_t}{p_0} \cdot (1+d)^{\frac{t}{\beta-1}}. \tag{5}$$

4.1.1 Consumption and Savings

Households optimize the present-value of current and future utility by setting their consumption and savings rates according to the conventional life-cycle hypothesis, represented by the following constrained-maximization problem:

$$Maximize\ u_{t=0} = \sum_{t=0}^{T_0} \frac{u_t(q_t)}{(1+d)^t}$$

$$s.t.\ \sum_{t=0}^{T_0} \frac{c_t^e}{(1+r)^t} = b_{t=0} + \sum_{t=0}^{K_0} \frac{y_t^e}{(1+r)^t}. \tag{6}$$

where b_0 denotes an initial wealth endowment, y_t^e is the expected nominal income earned in period t, and r denotes the market interest rate. We factor c_0 from the left side of the budget constraint from equation (6) to obtain

$$c_0 \cdot \sum_{t=0}^{T_0} \left(\frac{c_t^e}{c_0} \cdot \frac{1}{(1+r)^t} \right) = b_0 + \sum_{t=0}^{K_0} \frac{y_t^e}{(1+r)^t}, \tag{7}$$

and substitute equation (5) into $\frac{c_t^e}{c_0}$ to obtain

$$c_0 \cdot \sum_{t=0}^{T_0} \left(\frac{p_t^e}{p_0} \cdot \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t} \right) = b_0 + \sum_{t=0}^{K_0} \frac{y_t^e}{(1+r)^t}, \tag{8}$$

from which we obtain the optimal current consumption $\hat{q}_0$ and expenditure $\hat{c}_0$:

$$\hat{c}_0 = p_0 \hat{q}_0 = \frac{b_0 + \sum_{t=0}^{K_0} \frac{y_t^e}{(1+r)^t}}{\sum_{t=0}^{T_0} \left(\frac{p_t^e}{p_0} \cdot \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t} \right)}. \tag{9}$$

Since there is no money creation, we employ a simplifying assumption that expected prices are equal across time, which implies $\frac{p_t^e}{p_0} = 1$. To summarize, in each period, each household will borrow or save to achieve current consumption of

$$\hat{c}_0 = p_0 \hat{q}_0 = \frac{b_0 + \sum_{t=0}^{K_0} \frac{y_t^e}{(1+r)^t}}{\sum_{t=0}^{T_0} \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t}}. \tag{10}$$

The required savings transaction is

$$\hat{s}_0 = y_0 - \hat{c}_0, \tag{11}$$

where transactions are categorized as follows[2]:

$$\begin{Bmatrix} deposit: & \hat{s}_0 \geq 0 & \forall b_0 \\ withdraw: & \hat{s}_0 < 0 & \text{and } b_0 > 0 \\ loan: & \hat{s}_0 < 0 & \text{and } b_0 \leq 0 \end{Bmatrix}. \tag{12}$$

For retired households, who cannot earn income, equations (10) and (11) can respectively be simplified to

$$\hat{c}_0 = \frac{b_0}{\sum_{t=0}^{T_0} \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t}} \text{ and } \hat{s}_0 = -\hat{c}_0 \leq 0. \tag{13}$$

4.1.2 Firm Selection

When more than one firm engages in production, households must decide from which firm to purchase goods. In this model, each household randomly selects a firm each period, where firms with lower prices have higher probabilities of being selected. All households use the same selection rule, which is based on standard discrete-choice mechanics (McFadden 1974, Slepoy and Pryor 2002, Train 2003). Specifically, let ϕ denote the selection probability defined as

[2] The case of combined withdraw and loan occurs when $\hat{s}_0 < 0$ and $|\hat{s}_0| > b_0 > 0$.

$$\phi_{f_1} \equiv \Pr[\text{household } h \text{ selects firm } f_1] \equiv \frac{p_{f_1}^{\gamma}}{\sum_{f=0}^{F} p_f^{\gamma}}, \tag{14}$$

where $\gamma < -1$ is a constant, and p_f is the price charged by firm *f*. It can be shown that the relative probability of the household selecting firm f_1 over firm f_2 equals the scaled inverse of the firms' prices: $\dfrac{\phi_{f_1}}{\phi_{f_2}} = \left(\dfrac{p_{f_2}}{p_{f_1}}\right)^{|\gamma|}$.

4.1.3 Aggregate Supply and Demand

Since consumption of goods is the only source of utility for households, all employment-eligible households desire to work. Therefore, the aggregate household supply in the labor market is

$$L_t^S = E_t = \sum_{h=1}^{H} l_h \quad , l_h = \begin{Bmatrix} 1 \ \forall h \ni Age_0 \le Age_{retire} \\ 0 \ \forall h \ni Age_0 > Age_{retire} \end{Bmatrix}. \tag{15}$$

Aggregate net demand for money is derived from equation (11):

$$M^D = \sum_{h=1}^{H} \hat{s}_h(r, d_h, \beta_h, b_h, y, p). \tag{16}$$

Equation (10) implies that the aggregate household demand in the goods market is

$$Q^D = \sum_{h=1}^{H} \hat{q}_h = \sum_{h=1}^{H} \frac{\hat{c}_{0,h}}{p_0}. \tag{17}$$

4.2 Firms and Nash Equilibrium

For simplicity, wage rate *w* and productivity rate ρ in this exercise are fixed and equal across all firms. Firms search for the labor size and product price that maximize steady-state profits. Table 5 lists variables defined within the firms. A firm uses l_t units of labor to produce and supply q_t^S units of goods to the goods market using the production technology

$$q_t^S = \rho l_t . \tag{18}$$

Each firm sets its selling price p_t for goods, and earns production profit

$$\pi_t^{production} = \left(p_t q_t^{sold}\right) - wl_t \leq \left(p_t q_t^S\right) - wl_t = \left(p_t \rho - w\right) l_t . \tag{19}$$

Firms also participate as passive lenders in a banking market by making their cash reserves available for loans to households at the fixed market interest rate r . Interest earned from loans equals rA_t and provides a second source of profit to the firm. Summarizing, a firm's profit is defined by

$$\pi_t = \left(p_t q_t^{sold}\right) - wl_t + rA_t . \tag{20}$$

4.2.1 Employment and Production

Each firm can be either supply-constrained or demand-constrained. A firm is said to be supply-constrained if $q_f^S < q_f^D$, since under this condition the amount that the firm can sell is limited by the amount of labor it can hire; that is $q_f^{sold} = q_f^S \equiv \rho l_f$. A firm is said to be demand-constrained if $q_f^S > q_f^D$, since under this condition the amount that the firm can sell is limited by demand for its goods; that is $q_f^{sold} = q_f^D \equiv \phi_f Q^D$. Thus,

$$q_t^{sold} = \min\left(\phi_f Q_t^D ,\ \rho l_t\right). \tag{21}$$

Equations (20) and (21) yield a profitability condition for price:

$$\pi_t > 0 \quad \Leftrightarrow \quad p_t \ \geq \ \max\left(\frac{wl_t - rA_t}{\phi_f Q^D}, \frac{wl_t - rA_t}{\rho l_t}\right). \tag{22}$$

Supply-constrained firms hire employees to maximize profit $\pi_t = (p_t \rho - w) l_t + rA_t$, where

$$A_t \equiv R_t - X_t - wl_t, \tag{23}$$

which implies

$$\pi_t = (p_t\rho - w)l_t + r\left(R_t - X_t - wl_t\right). \tag{24}$$

If the firm's reserves R_t exceed current wage payments and loans, then $X_t > 0$ and the marginal profit from additional labor is $\frac{\partial \pi_t}{\partial l_t} = (p_t\rho - w)$. However, if current wage payments and loans exhaust the firm's reserves, then $X_t = 0$ and the firm must forego interest from loans in order to hire labor. In this case, the marginal profit from additional labor is $\frac{\partial \pi_t}{\partial l_t} = (p_t\rho - (1+r)w)$.

These results, in light of equation (18), provide the conditions under which profit-maximizing firms will try to obtain infinite profits by hiring infinite labor to produce infinite goods:

$$\hat{q}_t = \rho\hat{l}_t = \begin{Bmatrix} +\infty, & p_t > w/\rho & \text{and } X_t > 0 \\ 0, & p_t \le w/\rho & \text{and } X_t > 0 \\ +\infty, & p_t > (1+r)w/\rho & \text{and } X_t = 0 \\ 0, & p_t \le (1+r)w/\rho & \text{and } X_t = 0 \end{Bmatrix}. \tag{25}$$

Once the labor pool is exhausted, supply-constrained firms can only pursue profits via pricing, since by equation (24) profit is increasing with price: $\frac{\partial \pi_f}{\partial p_f} = \rho l_f > 0$. However, each firm's price is bounded by the market-share function in equation (14). Specifically, from

$$\left.\frac{\partial \phi_{f_0}}{\partial p_{f_0}}\right|_{\overline{p}_{f\neq f_0}} = \frac{\gamma\phi(1-\phi)}{p_{f_0}} < 0, \tag{26}$$

it follows that increases in a particular firm's price will reduce ϕ_f until $\rho l_t \ge \phi_f Q^D$ at which time the firm becomes demand-constrained rather than supply-constrained. By equations (21) and (22), the demand-constrained firm has profit

$$\pi_t = p_t \phi_t Q^D - wl_t + rA_t . \qquad (27)$$

This equation is used later to derive the Nash equilibrium price distribution.

4.2.2 *Aggregate Supply and Demand*

Equation (18) implies that the aggregate supply of goods is

$$Q_t^S = \sum_{f=1}^{F} \rho l_{f,t} = \rho E_t . \qquad (28)$$

Equations (21) and (25) imply that firms' aggregate demand for labor is

$$L^D = \sum_{f=1}^{F} \hat{l}_f = \sum_{f=1}^{F} \frac{\hat{q}_f}{\rho} = \{0, +\infty\}. \qquad (29)$$

We derive the firms' aggregate supply of money from Table 5 and equation (23):

$$M^S = \sum_{f=1}^{F} \left[R_f - wl_f\right]. \qquad (30)$$

4.3 Market Equilibrium

The necessary conditions for market clearing are

$$Q^S = Q^D ,\ L^S = L^D \text{ and } M^S = M^D . \qquad (31)$$

4.3.1 *Nash Equilibrium Prices*

Under the discrete choice defined in equation (14), equilibrium prices will vary across firms as the scaled inverse of labor shares. To show this, we first note from equation (11) that $\sum_{h=1}^{H} \hat{c}_{h,t} = \sum_{h=1}^{H} y_{h,t} - \sum_{h=1}^{H} \hat{s}_{h,t}$, which implies $Y_t = S_t + \sum_{h=1}^{H} \hat{c}_{h,t} = \theta_{S,t} Y_t + \sum_{h=1}^{H} \hat{c}_{h,t}$, where $\theta_{S,t} \equiv \frac{S_t}{Y_t}$.

Since $Y_t = wL_t$, we have $\sum_{h=1}^{H} \hat{c}_{h,t} = (1-\theta_{S,t}) wL_t$. Substituting into equation (17) yields

$$Q_t^D = \frac{(1-\theta_{S,t})wL_t}{p_t}. \tag{32}$$

By definition, the market-clearing condition for an individual firm f is $q_f^D = q_f^S$. By equations (18) and (21), we have

$$q_f^D = q_f^S \quad \Leftrightarrow \quad \phi_f Q^D = \rho l_f \quad \Leftrightarrow \quad \phi_f = \frac{\rho l_f}{Q^D}. \tag{33}$$

Substituting equation (32) into (33) yields $\phi_f = \dfrac{\rho l_t p_t}{(1-\theta_{S,t})wL_t}$. Incorporating equation (14) and Table 5 provides the Nash equilibrium price, given by

$$q_f^D = q_f^S \quad \Leftrightarrow \quad p_f^* = \left[\frac{\rho \cdot g_t \cdot k_t \cdot \sigma_{f,t}}{w}\right]^{\frac{1}{\gamma-1}}, \tag{34}$$

where $k_t = \sum_{f=1}^{F} p_{f,t}^{\gamma}$ and $g_t = \dfrac{1}{1-\theta_t}$. That is, given its labor share $\sigma_{f,t}$ and the other firms' prices k_t, firm f cannot benefit by charging any price different from p_f^*.[3] The following partial derivative shows that equilibrium prices vary as the scaled inverse of labor share:

$$q_f^D = q_f^S \quad \Leftrightarrow \quad \frac{\partial p_f^*}{\partial \sigma_f} = \left(\tfrac{1}{\gamma-1}\right)\left[\frac{\rho \cdot g_t \cdot k_t}{w}\right]^{\frac{1}{\gamma-1}} \sigma^{\frac{\gamma}{1-\gamma}} < 0. \tag{35}$$

Equations (34) and (35) imply that smaller firms are able to settle at higher prices; the reason is that smaller firms produce fewer goods and therefore require a smaller market share to maximize profit.

[3] To clarify, our model assumes that wages are fixed. Therefore, firms search for their optimal labor, but they cannot compete for labor by offering higher wages. This assumption introduces rigidity into the labor market, leading us to define Nash equilibrium in terms of goods prices alone. Although the firms are homogenous in all other respects, they are made heterogeneous by their relative allocations of labor. A more general model might relax this rigidity in the labor market, allowing one to define Nash equilibrium in terms of goods price and labor (p_f^*, l_f^*), in which case the firms would be truly homogenous and the Nash equilibrium would specify equal prices and labor shares.

4.3.2 Banking

The banking industry clears with no tradeoffs between wages and loans when money supplied by firms covers the money demanded by households. Equations (16) and (30) imply the clearing condition,

$$M_t^D \le M_t^S \quad \Leftrightarrow \quad \sum_{h=1}^{H} \hat{s}_{h,t} \le \sum_{f=1}^{F} \left[R_{f,t} - wl_{f,t} \right], \tag{36}$$

which by Table 3 implies

$$M^D \le M^S \quad \Leftrightarrow \quad S_t + Y_t \le \sum_{f=1}^{F} \left[R_f \right]. \tag{37}$$

Therefore, the banking market clears with no wage-loan tradeoffs as long as firms' total reserves exceed households' incomes and net savings. Note that Y_t is non-negative, but S_t is negative if households are net borrowers.

5 Simulation

This section presents a computer simulation of the prescribed model. The simulation uses independent computer programs to simulate households, firms, and a bank. The computer programs interact by passing messages in a discrete-time environment. We first describe basic mechanics for the handling of time and messaging. Next, we describe the agents' decision rules, assign input parameters, and calculate equilibrium conditions. Finally, we present simulation results and demonstrate that the simulation converges to the expected Nash equilibrium.

5.1 Mechanics

We implemented this simulation in an modified version of a Sandia agent-based modeling package called Aspen. See Basu et al. (1998) for details on the structure and uses of Aspen.

All agents maintain private information and interact via message passing. At the start of each time period, each agent conducts an assessment of its state variables and formulates a list of objectives and action items. Action items generally involve transactions with other agents. Some transactions require iterative communication. For example, a household seeking employment must send a job application to a firm, which responds with an offer or rejection. Subsequently, the household must accept or reject any job offers.

To allow for proper assessment and interaction, each time period is divided into an assessment step in which all agents set their objectives and action items, followed by several messaging steps to allow agents to complete transactions during the time period.

5.2 Agents

The economic actors described in the prescribed model are represented by autonomous agents who store private information and make economic decisions. Firms and households make decisions in pursuit of clearly-defined economic objectives. Households make their consumption decisions based on a single equation, whereas firms use more complex decision rules by which they search, remember, and react. However, none of the agents in this exercise are *adaptive* in the sense of employing genetic algorithms or other means of learning over time. Thus, this exercise demonstrates that even simple agents with limited information can discover Nash equilibrium prices in a competitive market.

Table 6 specifies input parameters for the simulation. The household parameters are used to calculate the household optima in Table 7, and the firm parameters are used to calculate the firms' Nash equilibrium condition in Table 8.

5.2.1 Bank

There is a single bank agent that serves as an intermediary between households and firms. The bank holds the firms' excess reserves, which are made available as loans to households. The bank establishes an account for each household. Account balances can be positive or negative. Positive balances represent deposits, and accrue interest for the household at market interest rate *r*. Negative balances represent loans, and accrue interest for the bank at market interest rate *r*.

For simplicity, we assume that the bank applies the same interest rate to both loans and deposits, which implies that the bank, and therefore firms, should only engage in banking if households are net borrowers in the aggregate. A more general implementation of banking would allow for a spread between lending and saving rates, but is beyond the scope and purpose of this exercise. We ensure integrity in the banking portion of the simulation by specifying a distribution of households whose aggregate savings profile is negative, and by setting initial reserves to satisfy equation (37).

5.2.2 Households and Calculated Optima

Households maintain the variables defined in Table 4. At the start of each time period, each household makes two primary assessments. First, if an employment-eligible (career) household is unemployed, then it sends a job application to a firm. Second, all households calculate their target consumption and savings according to equations (10) and (11). Equation (11) is calculated in part using income in the current period, which is known:

$$y_{h,t=0} = \left\{ \begin{array}{ll} 0, & \text{if unemployed} \\ w, & \text{if employed} \end{array} \right\}. \tag{38}$$

However, the calculation also requires career households to make assumptions regarding future income. For simplicity, households in this exercise optimistically assume they will be employed in all future employment-eligible periods; that is, $y^e_{h,t} = w \quad \forall t \ni 0 < t < K_0$. Once the consumption expenditure is determined, the household selects a firm from which to purchase goods according to equation (14), sends a purchase order to the selected firm, and wires a bank transaction in accordance with equation (11) for a deposit, withdraw, or loan.

We instantiate households with a uniform age distribution and identical input parameters, specified in Table 6. Each household is comprised of a single individual who becomes employment-eligible at age 20, retires after age 60, and dies after age 80. There are 5 time periods per year, resulting in 301 periods per each individual's lifespan. The simulation instantiates 301 households with a uniform age distribution, so there is exactly one household associated with each period in the life cycle. Each household has one descendent who becomes employment-eligible when its parent dies, and which inherits any remaining debts or bank deposits. Under this framework, the age distribution is cyclical and corresponds to the initial age distribution. Additionally, all households have the same discount rate and consumption elasticity. The combined assumptions of uniform age distribution and equal discount rate and consumption elasticity greatly simplify calculations of the expected aggregate profile: that is, summing the periods of the discounted cash-flow profile for a single household provides the expected aggregate profile for the population of households. Since all career households desire jobs, the expected aggregate employment level, *E*, equals the number of career households. Since households do not expect price changes, their consumption expenditures and savings levels are independent of goods prices. The aggregate optimal consumption and savings were calculated in a spreadsheet based on discount rate and consumption elasticity and listed in Table 7.

5.2.3 Households and Unexpected Unemployment

In this exercise, all households have a fixed positive discount rate, resulting in greater planned consumption in earlier years. Figure A-1 (see appendix A) shows the optimal planned consumption expenditure for a household calculated for the parameters specified in Table 6. Under these parameters, younger households achieve the optimal consumption by borrowing against future earnings. Figure A-2 shows the corresponding planned bank transactions required to achieve the consumption schedule shown in Figure A-1. We see that each household will borrow loans through age 38, make loan payments and deposits from age 38 to 60, and make withdraws after retirement at age 60.

In this exercise, households never anticipate future unemployment, and when it occurs the unemployed household never expects it to continue beyond the current period. Thus, unemployed households do not significantly reduce current consumption, but rather borrow significant loans in order to finance their consumption while unemployed. Consider a hypothetical situation in which a household is unemployed for three periods (7 months) at ages 30, 40, and 50. In this exercise, the household's response to such periods of unexpected unemployment would manifest as spikes of borrowing (i.e. unemployment loans), shown in the revised bank transactions in Figure A-2. Figure A-3 shows the impact of these unemployment loans on lifetime bank balances. Each loan reduces the bank balance, and revised balances remain below planned balances for the remainder of the household's lifespan. Figure A-4 shows the impact of unemployment loans on lifetime consumption expenditure, which is also revised downward and remains below planned consumption for the remainder of the lifespan.

To summarize, unemployment in this simulation moves consumption and savings to a lower schedule for the duration of the a household's lifespan. As explained below, there will often be some unemployment in the simulation resulting from each firm's search for its optimum employment level. Thus, we expect to observe average savings and consumption in the simulation that are lower than the planned optima listed in Table 7.

5.2.4 Firms and Nash Equilibrium

Firms maintain the variables defined in Table 5. At the start of each time period, each firm makes two primary decisions: (1) whether to increase or decrease its labor force, and (2) whether to increase or decrease its price for goods. Unlike households, firms do not have a simple equation to dictate their decisions, but must search for optimal labor and prices using simple algorithms.

Firms make their labor decision by assessing a running record of profits[4] for the previous V periods, and altering their strategies for *scaling* their labor forces. At the start of each period, each firm calculates *recent* profits $\sum_{t=1}^{V/2} \pi_{-t}$ and *earlier* profits $\sum_{t=(V/2)+1}^{V} \pi_{-t}$. The firm also knows its scaling strategy from the previous period, represented by a scaling variable $\delta_{-1} = \frac{l^*_{-1}}{l^*_{-2}} \in (0,2]$, where $\delta_t \in (0,1)$ denotes "layoffs" in period t, and $\delta_{-1} \in (1,2]$ denotes attempts to hire new employees in period t. Each firm compares recent and earlier profits. If recent profits exceed earlier profits, then the firm re-adopts its labor strategy from the previous period to define its new desired number of employees: $l^*_0 \equiv \delta_0 \cdot l^*_{-1}$, where $\delta_0 \equiv \delta_{-1}$. Otherwise, the firm reverses its labor strategy by setting its new desired number of employees to the level that obtained the highest profits in the previous V periods: $l^*_t \equiv \max\{l_{-t}\}_{t=1}^{V}$. In this latter case, the firm notes its

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[4] For purposes of generality, firms in this simulation search for the labor-scale that maximizes *real* profits, defined as $\chi \equiv \pi/p$, which remains consistent with the conditions of equation (25).

reversal by resetting its labor-strategy: $\delta_0 = \frac{l_0^*}{l_{-1}^*}$. If a firm converges to the same number of employees for V consecutive periods, then the firm tests its current scale by randomly either increasing or decreasing its labor force by one employee. We restrict firms in this exercise to desire at least one employee: $l_t^* \geq 1$.

Firms set prices by assessing a running record of profits[5] for the previous V periods, and altering their *price-adjustment* strategies. Each firm employs the same algorithm for price as it does for labor-scale, except for two minor distinctions. First, prices are real numbers instead of integers. Therefore, if a firm converges to the same price for V consecutive periods, then the firm tests its current price by randomly increasing or decreasing price by one *percent*, rather than by one *unit*. Second, if $p_t > 0$ and $q_t^{sold} = 0$, then the firm reduces price by one percent.

Equation (34) defines the Nash equilibrium for prices. We obtain a testable proposition for Nash equilibrium by rewriting equation (34) in natural logarithms as follows:

$$\ln(p_f) = \alpha + \eta \ln(g_t) + \eta \ln(k_t) + \eta \ln(\sigma_{f,t}), \tag{39}$$

where $\eta \equiv \frac{1}{\gamma - 1}$ and $\alpha \equiv \eta \ln\left(\frac{\rho}{w}\right)$. Since $\gamma = -2$, the expected price-labor coefficient is $E[\eta] = -1/3$, as indicated in Table 8. Also, since $\rho = 2$ and $w = 50$, the expected wage-productivity constant is $E[\alpha] = \eta \ln\left(\frac{\rho}{w}\right) \approx \frac{3.22}{3} \approx 1.07$.

We test the relationship in equation (39) by exporting price, labor and savings-ratio data from the simulation to estimate the following equation:

[5] For simplicity, firms in this simulation search for the price that maximizes revenue in the goods market, rather than profit. To show that this is a valid substitution, let $R \equiv p\phi Q^D$ denote revenue in equation (27), and note that $\partial \pi / \partial p = \partial R / \partial p$.

$$\ln(p_f) = \beta_0 + \beta_1 \ln(g_t) + \beta_1 \ln(k_t) + \beta_1 \ln(\sigma_{f,t}) . \qquad (40)$$

The test will confirm Nash equilibrium prices if $\beta_0 = E[\alpha]$ and $\beta_1 = E[\eta]$. To ensure variation in labor share, we explicitly vary the firms' desired number of employees in the first year so that initial labor-share varies from 11% to 27% across the *F=5* firms.

5.3 Simulation Results

Simulation results are presented as time-series plots in the Appendices B and C. Figure B-1 (appendix B) plots total employment, which fluctuates near the optimal level of 201, but reveals periodic layoffs due to the firms' search algorithms. As explained in section 5.2.3, such layoffs should result in average consumption and bank balances that are lower than the calculated optima listed in Table 7, which indeed occurs. Figures C-1a and C-1b (appendix C) show that average consumption expenditure and bank balance fluctuate near, but below, their calculated optima. Figures C-2a and C-2b show this same result specifically for career households, and figures C-3a and C-3b show this same result specifically for retired households.

Figure B-2 (appendix B) reveals that the average price of goods fluctuates within the range of 20 to 25. By equation (34), we do not expect a single market equilibrium price, but rather a range of prices that vary across firms with respect to labor share and other factors. Therefore, to test for Nash equilibrium, we exported a record of data for each of the 5 firms every 100 periods from period 1000 to 3000, resulting in 105 records of data. Each record includes data for the firm's price, the firm's labor share, the sum of scaled prices: $\sum_{f=1}^{F} p_f^{\gamma}$, and the savings-income variable: $\frac{1}{1-\theta_t}$. Equation (34) implies an inverse relationship between prices and labor share, which is revealed in Figure 1. Each point in the graph is labeled with the ID number of the corresponding firm.

We formally test for Nash equilibrium by estimating equation (40) using the exported data. We first define the following regression equation:

$$\ln(p_f) = \beta_0 + \beta_g \ln(g_t) + \beta_k \ln(k_t) + \beta_\sigma \ln(\sigma_{f,t}) . \tag{41}$$

We obtain equation (40) from (41) by imposing the following constraint: $\beta_1 = \beta_g = \beta_k = \beta_\sigma$. We imposed this constraint and estimated equation (40) using the constrained linear regression command in a widely accepted statistical software package called STATA (see StataCorp 2003). The results are shown in Figure 2. The regression provides the following 95% confidence intervals: $\hat{\beta}_0 \equiv estimate(\alpha) \in [0.852, 1.189]$ and $\hat{\beta}_1 \equiv estimate(\eta) \in [-0.378, -0.322]$, which contain the expected values listed in Table 8, and thereby confirm that the simulation converges rather precisely to its Nash equilibrium.

6 Remarks

We find that firms with little information and simplistic decision processes "discover" their Nash equilibrium prices. The exercise achieves minimum success under the criteria of Table 1. Table 9 emphasizes the importance of convergence by highlighting the information that is unavailable to the firms in the course of their discovery. The firms converge despite relative ignorance and a reliance on extremely rudimentary search algorithms. Thus, the simulation supports the robustness and validity of the analytical model.

Positive economics (Friedman 1953) conventionally involves the use of accepted axioms to formulate a theory that explains observed behavior and forecasts the response to changing economic conditions. The exercise above extends this approach: by verifying a baseline simulation against an conventional theory, we are free to expand our investigation to more interesting and complex exercises in a framework that is less encumbered by simplifying assumptions.

For homeland-security problems, this model provides a dynamic foundation for modeling economic and financial responses. For example, agent rules can be designed in accordance with various explicit models of confidence (Batchelor and Dua 1998, Bram and Ludvigson 1998, Desroches and Gosselin 2002, and Garner 2002) to explore potential responses to hypothetical economic shocks.

The simulation allows for general-purpose extensions to explore current problems in economics, such as the role of leisure in consumption profiles (Heckman 1974, Becker and Ghez 1975, Bullard and Feigenbaum 2004), precautionary savings (Hubbard et al 1994, Carroll 1997, Abel 1985, 2001, Poterba 2001, and Wang 2004), and trade between multiple interactive economies (Sayan and Uyar 2002).

Appendix A: Impacts of Unexpected Unemployment

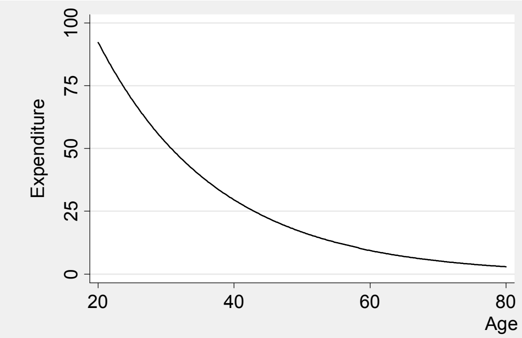

Figure A-1. Planned Lifetime Consumption

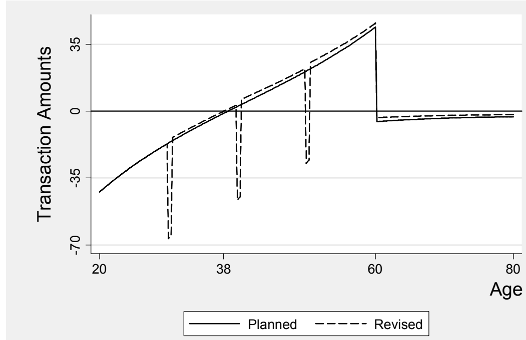

Figure A-2. Lifetime Bank Transactions

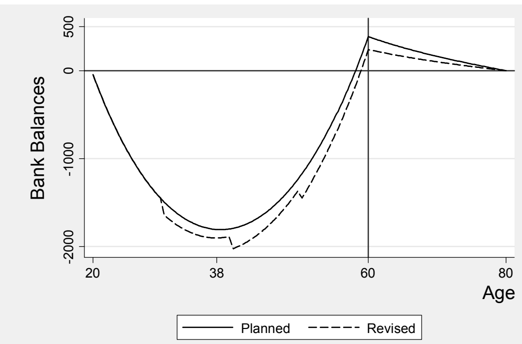

Figure A-3. Lifetime Bank Balances

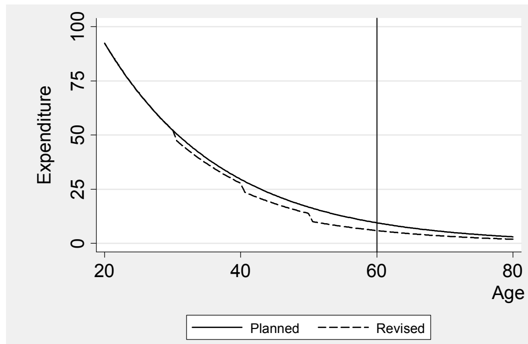

Figure A-4. Lifetime Consumption

Appendix B: Simulation Results for Market Outcomes

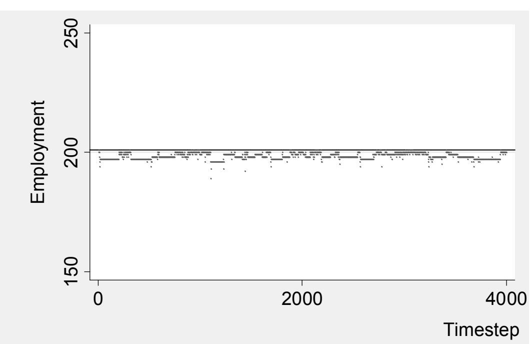

Figure B-1. Total Employment

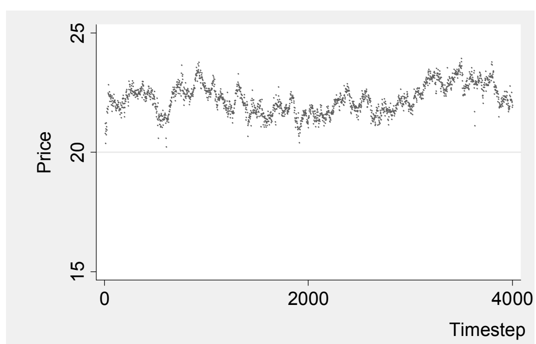

Figure B-2. Average Goods Price

Appendix C: Simulation Results for Household Variables

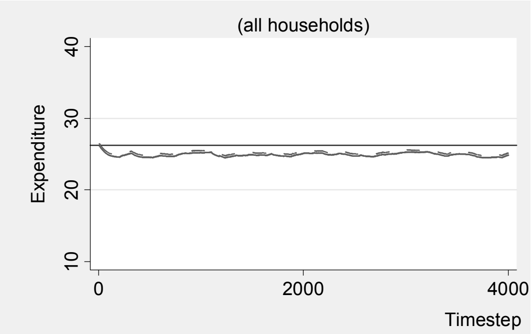

Figure C-1a. Average Expenditure

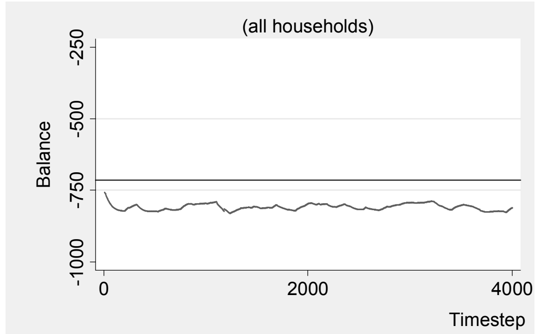

Figure C-1b. Average Bank Balance

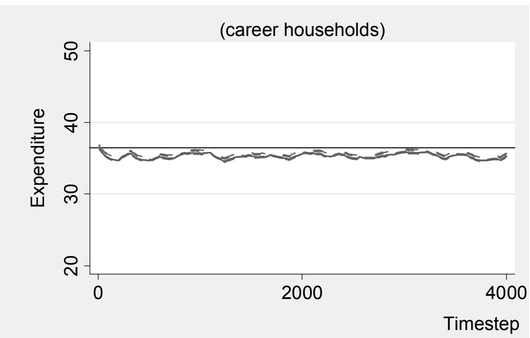

Figure C-2a. Average Expenditure

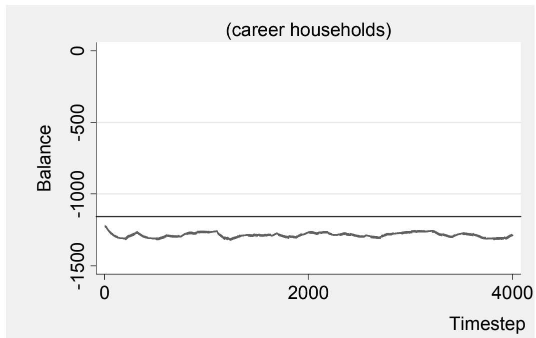

Figure C-2b. Average Bank Balance

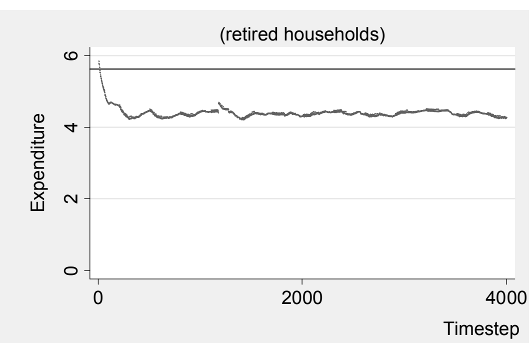

Figure C-3a. Average Expenditure

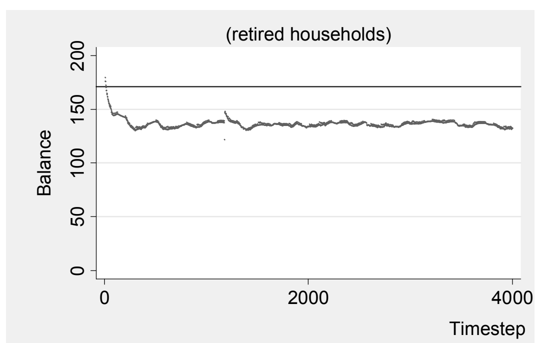

Figure C-3b. Average Bank Balance

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Table 1. Exercise Assessment Criteria

Success Rating	Criteria
Minimum	The simulation converges to the analytically derived solution, and allows for meaningful (non-trivial) observations of the agent characteristics and environmental conditions that allow for convergence.
Moderate	An analytically derived solution exists, but the simulation converges to an alternate outcome for non-obvious reasons, allowing for postulates and subsequent investigation.
High	An analytically derived solution exists, but the simulation converges to an alternate outcome for non-obvious reasons. Subsequently, one or more simulations can be found that do converge to the analytically derived solution. A collection of conditions, rules, or interaction criteria are identified that lead to alternate outcomes. The original theory is extended or modified to encompass the new findings.
Experimental	Two models are formed. The initial model has a known closed-form solution and a corresponding simulation that converges to that solution. An extended model is formed, which (1) is a generalization of the initial model, (2) has no closed-form solution, and (3) has a corresponding simulation that converges an alternate outcome. The outcomes of the respective simulations are compared to provide a proxy comparison of the respective models.

Table 2. Constant Parameters

ρ	≡ units of goods produced by unit labor per period
β	≡ consumption-elasticity of household utility
γ	≡ price-sensitivity exponent
λ	≡ periods per year
w	≡ wage rate
r	≡ market annual interest rate for bank loans and deposits
H	≡ number of households
F	≡ number of firms
$Age_{\min}$	≡ minimum employment age
Age_{retire}	≡ mandatory retirement age
$Age_{\max}$	≡ age of death

Table 3. Market Variables

p_t	≡ price per unit goods; $p_t \geq 0 \in$ {*reals*}
q_t	≡ units of goods; $q_t \geq 0 \in$ {*reals*}
l_t	≡ units of labor; $l_t \geq 0 \in$ {*integers*}
E_t	≡ number of employed households
L_t	≡ aggregate units of labor
Q_t	≡ aggregate units of goods
Y_t	≡ aggregate nominal wages (payrolls)
C_t	≡ aggregate household consumption expediture
S_t	≡ aggregate household savings = deposits - debts
$\theta_{S,t}$	≡ aggregate household savings rate
M_t	≡ money

Table 4. Household Variables

d_h	≡ household *h*'s fixed discount rate for all periods t,
$u_{h,t}$	≡ household *h*'s utility,
$\phi_{f_0,t}$	≡ Pr[any household purchases goods for firm f_0],
$y_{h,t}$	≡ household *h*'s income,
$c_{h,t}$	≡ household *h*'s consumption expenditure,
$s_{h,t}$	≡ household *h*'s increment savings ≡ $y_{h,t} - c_{h,t}$, and
$b_{h,t}$	≡ household *h*'s wealth ≡ cash + deposits – debts.

Table 5. Firm Variables

$p_{f,t}$	≡ goods price offered by firm *f*
$l_{f,t}$	≡ units of labor employed at firm *f*
$\pi_{f,t}$	≡ nominal profit ≡ revenue – cost
$\chi_{f,t}$	≡ real profit
$\sigma_{f,t}$	≡ firm *f*'s labor share
$R_{f,t}$	≡ firm *f*'s money reserves = cash + payroll + deposits - loans
$A_{f,t}$	≡ firm *f*'s net loans to households = loans - deposits
$X_{f,t}$	≡ firm *f*'s cash holding

Table 6. Simulation Parameters

	Parameters	Symbol	Value
Global			
	Number of time periods	-	4000
	Number periods per year	λ	5
	Wage rate	w	50
	Market interest rate	r	5.0%
Households			
	Number of households	H	301
	Consumption-elasticity of utility	β	0.3
	Price-sensitivity exponent	γ	-2
	Discount rate	d	4.0%
	Minimum employment age	$Age_{\min}$	20
	Mandatory retirement age	Age_{retire}	60
	Expiration age	$Age_{\max}$	80
	Age distribution (uniform)	$\{Age_{h,t=0}\}_H$	~[20,80]
	Employment-eligible households	E	201
Firms			
	Number of firms	F	5
	Productivity rate	ρ	2
	Initial reserves	$R_{f,t=0}$	$200K

Table 7. Household Optima

Variable	Symbol	Value
Aggregate employment	E	201
Average consumption expenditure	$\overline{C}$	$26.22
- Career households		$36.46
- Retired households		$5.62
Average bank balances (deposits – debts)	$\overline{B}$	($717)
- Career households		($1159)
- Retired households		$171

Table 8. Nash Equilibrium

Variable	Symbol	Expected Value
Price-labor coefficient	η	-1/3
Wage-productivity constant	α	1.07

Table 9. Information Unavailable to Each Firm

Prices charged by other firms
Selection rule used by households; see equation (14)
Number of consumers (i.e. market share)
Number of laborers (i.e. labor share)
Customer churn/retention rates
Nash equilibrium conditions

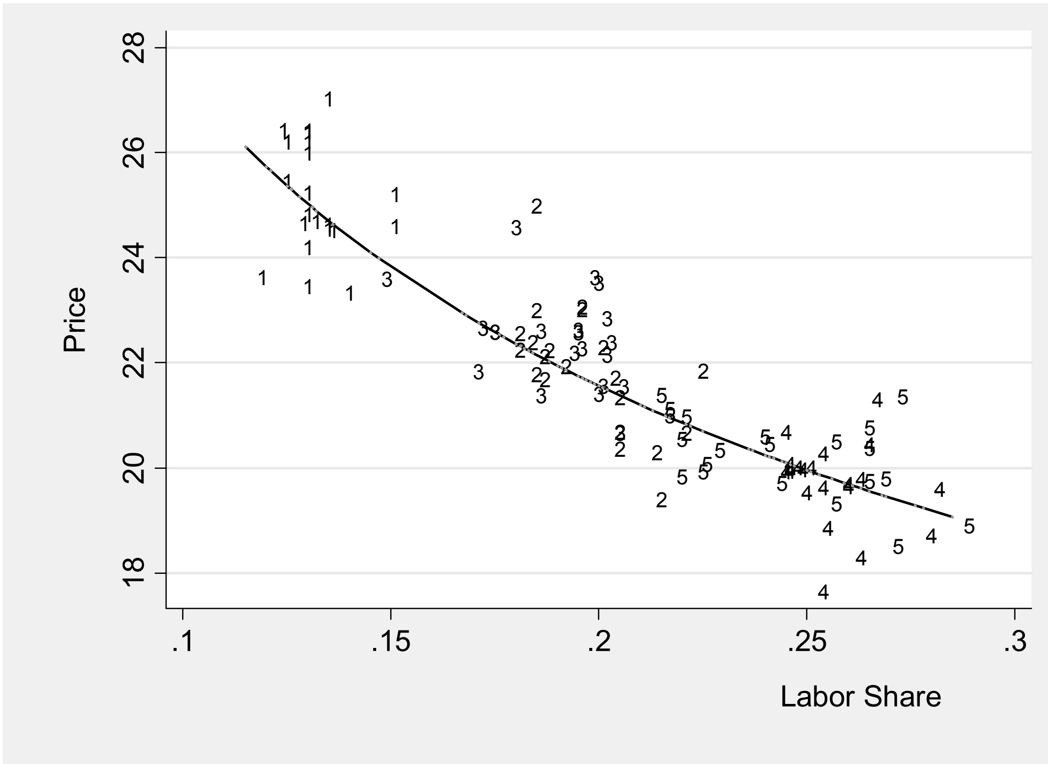

Figure 1

```
Constrained linear regression                       No. of obs =    105
                                                    F( 1, 103) =    585
                                                    Prob > F   =   .000
----------------------------------------------------------------------
       Ln(p) |      Coef.         t    P>|t|     [95% Conf. Interval]
-------------+--------------------------------------------------------
       Ln(g) |     -0.349    -24.20   0.000     [ -0.378  , -0.321 ]
       Ln(k) |     -0.349    -24.20   0.000     [ -0.378  , -0.321 ]
       Ln(σ) |     -0.349    -24.20   0.000     [ -0.378  , -0.321 ]
         α   |      1.020     11.98   0.000     [   .852  ,  1.189 ]
```

Figure 2. Estimated Relationship between Price and Labor-Share